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APPLICATION OF BANACH LIMITS TO THE STUDY OF SETS OF INTEGERS

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ABSTRACT

The Kamae and Mendes France version of the Van der Corput equidistribution theorem is extended further to summability methods different from Cesàro summability and groups different from the circle. The theorem is shown to follow naturally from consideration of Banach limits and spectral theory.

Introduction

Kamae and Mendes France [10] call a set of integers $H \subset \mathbf{N}$ a *Van der Corput set* iff whenever $\{x_n\}$ is a sequence in $\mathbf{T} = [0, 1)$ for which the differences $\{x_{n+h} - x_n\}_n$ are equidistributed modulo 1 for all $h \in H$, necessarily $\{x_n\}$ is equidistributed itself. This notion is motivated by Van der Corput's difference theorem, which states, in effect, that $\mathbf{N}$ is a Van der Corput (VDC) set. In this paper we examine collections of sets of integers related to the collection of VDC sets, using Banach limits as our main tool. In Section 1 we characterize FC^+-sets — sets of integers forcing continuity of positive measures. In Section 2 we summarize some relevant results from the theory of summability methods.

In Section 3 we extend the result of Kamae and Mendes France that any FC^+-set is a VDC set, to other summability methods. In Section 4 we give two quantitative versions of the Van der Corput theorem, following the fundamental work of Ruzsa (see Ruzsa [18], [19]).

We also consider VDC sets with respect to compact groups different from $\mathbf{T}$.

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1. Sets of integers forcing continuity of positive measures

DEFINITION. Let **T** be the circle, which we identify with [0, 1). Denote by **P(T)** the set of probability measures on **T**. A set H of positive integers will be called an FC$^+$-set if for any $\mu \in \mathbf{P}(\mathbf{T})$

$$(\text{for all } h \in H, \hat{\mu}(h) = 0) \Rightarrow \mu\{0\} = 0$$

where

$$\hat{\mu}(h) = \int_0^1 \exp(-2\pi i h\theta) d\mu(\theta).$$

In the sequel we will show that this definition, due to Y. Katznelson, unifies results of Furstenberg [8] and Kamae and Mendes France [10].

EXAMPLE I. For $J \subset \mathbf{N}$ let $J - J = \{m - n \mid m, n \in J, m > n\}$ be the (positive) difference set of J. If J is infinite, $J - J$ is an FC$^+$-set (Kamae and Mendes France [10]).

In fact, the following stronger statement is easily proved (Bertrand-Mathis [1]):

$$\forall \mu \in \mathbf{P}(\mathbf{T}), \qquad \mu\{0\} \leqq \limsup_{\substack{h \to \infty \\ h \in J - J}} \mathrm{Re}\{\hat{\mu}(h)\}.$$

EXAMPLE II. Assume $H \subset \mathbf{N}$ has the following property: For each $k \in \mathbf{N}$, the set $H \cap k\mathbf{N}$ contains an infinite sequence $\{b_j^{(k)} \mid j \geqq 1\}$ such that for all irrational α, the sequence $\{b_j^{(k)} \cdot \alpha \mid j \geqq 1\}$ is equidistributed modulo 1. Then H is an FC$^+$-set.

This was proved by Kamae and Mendes France [10] and, in a stronger form, by Bertrand-Mathis [1].

For any polynomial $P: \mathbf{Z} \to \mathbf{Z}$ which does not vanish identically and has a root modulo k for each $k \in \mathbf{N}$, the set $P(\mathbf{Z}) \cap \mathbf{N}$ of positive values of P satisfies the property of Example II and is therefore an FC$^+$-set.

The following theorem characterizes FC$^+$-sets using isometries.

THEOREM 1.1. *Let $H \subset \mathbf{N}$. The following conditions are equivalent:*

(i) *H is an* FC$^+$*-set.*

(ii) *For any isometry U of an inner product space Ω, if a vector $\omega \in \Omega$ satisfies $U^h\omega \perp \omega$ for all $h \in H$, then $\omega \perp \mathrm{Ker}(U - I)$.*

Before proving the theorem, we note that its conclusion can be strengthened:

COROLLARY 1.2. *Let $H \subset \mathbf{N}$ be an* FC$^+$*-set and let U be an isometry of* *an inner product space* Ω. *If* $\omega \in \Omega$ *satisfies* $U^h\omega \perp \omega$ *for all* $h \in H$, *then* $\omega \perp \mathrm{Ker}(U - \zeta I)$ *for all* $\zeta \in \mathbf{C}$ *such that* $|\zeta| = 1$.

PROOF OF COROLLARY 1.2. Apply Theorem 1.1 to the isometry $\bar{\zeta}U$ of Ω. □

We shall prove a quantitative version of Theorem 1.1. Following Ruzsa [18], define for any $H \subset \mathbf{N}$:

$$\lambda(H) = \sup\{\mu\{0\} \mid \mu \in \mathbf{P}(\mathbf{T}), \hat{\mu}(H) = \{0\}\}$$

where $\hat{\mu}(H) = \{\hat{\mu}(h) \mid h \in H\}$. Note that $H \subset \mathbf{N}$ is an FC$^+$-set iff $\lambda(H) = 0$. Also denote:

$$\lambda_1(H) = \sup\{|\langle \omega, z\rangle|^2 \mid \omega \in \Omega,\ \forall h \in H\ U^h\omega \perp \omega,\ z \in \mathrm{Ker}(U - I),\ \|\omega\| = \|z\| = 1\}$$

where Ω runs over all inner product spaces and U over all isometries of Ω.

Notice that for $H \subset \mathbf{N}$, $\lambda_1(H) = 0$ iff H satisfies statement (ii) in Theorem 1.1. The following theorem clearly extends Theorem 1.1:

THEOREM 1.3. *For all* $H \subset \mathbf{N}$, $\lambda(H) = \lambda_1(H)$.

PROOF. (a) $\lambda \leqq \lambda_1$. Let $\mu \in \mathbf{P}(\mathbf{T})$ satisfy $\hat{\mu}(H) = 0$. We define $\Omega = L^2(\mu)$ and for each $f \in \Omega$, $Uf(\theta) = e^{-2\pi i\theta}f(\theta)$. U is an isometry of Ω and the function

$$z(\theta) = \begin{cases} [\mu\{0\}]^{-1/2}, & \theta = 0 \\ 0, & \theta \neq 0 \end{cases}$$

satisfies $z \in \mathrm{Ker}(U - I)$, $\|z\| = 1$.

Letting $\omega \equiv \mathbf{1}$ be a constant, we have $\forall h \in H$ $\langle U^h\omega, \omega\rangle = \hat{\mu}(h) = 0$ and therefore

$$\lambda_1 \geqq |\langle \omega, z\rangle|^2 = \left|\int z(\theta)d\mu(\theta)\right|^2 = \mu\{0\}.$$

(b) $\lambda_1 \leqq \lambda$. Let Ω be an inner product space, $U: \Omega \to \Omega$ an isometry, $z \in \mathrm{Ker}(U - I)$ and $\omega \in \Omega$ so that

$$\forall h \in H, \qquad U^h\omega \perp \omega \quad \text{and} \quad \|\omega\| = \|z\| = 1.$$

We must show that $|\langle \omega, z\rangle|^2 \leqq \lambda(H)$. Without loss of generality we can assume that Ω is a Hilbert space.

Let μ_ω be the spectral measure corresponding to ω:

$$\forall n \geqq 0, \qquad \hat{\mu}_\omega(n) = \langle U^n\omega, \omega\rangle.$$

$\mu_\omega \in \mathbf{P}(\mathbf{T})$ since $\hat{\mu}_\omega(0) = \langle \omega, \omega \rangle = 1$.

Let $\mathbf{M} = \mathrm{Ker}(U - I)$ and write $\omega = \omega_1 + \omega_2$ with $\omega_1 \in \mathbf{M}$, $\omega_2 \perp \mathbf{M}$.

$\mathbf{M}$, $\mathbf{M}^\perp$ are U-invariant spaces, so

$$\forall h \geqq 0, \qquad \langle U^n\omega, \omega \rangle = \langle U^n\omega_1, \omega_1 \rangle + \langle U^n\omega_2, \omega_2 \rangle.$$

Therefore $\mu_\omega = \mu_{\omega_1} + \mu_{\omega_2}$.

Now $\hat{\mu}_{\omega_1}(n) = \langle U^n\omega_1, \omega_1 \rangle = \| \omega_1 \|^2$ so $\mu_{\omega_1} = \| \omega_1 \|^2 \cdot \delta_0$, where δ_0 is the point mass at 0. $\hat{\mu}_\omega(H) = 0$ by the definition of μ_ω. Hence

$$\lambda \geqq \mu_\omega\{0\} \geqq \| \omega_1 \|^2 \geqq |\langle \omega_1, z \rangle|^2 = |\langle \omega, z \rangle|^2. \qquad \square$$

Remark. $\mu \to \mu\{0\}$ is an upper semicontinuous function on $\mathbf{P}(\mathbf{T})$, so the weak-* compactness of $\mathbf{P}(\mathbf{T})$ implies

$$\lambda(H) = \max\{\mu\{0\} \mid \mu \in \mathbf{P}(\mathbf{T}), \hat{\mu}(H) = 0\}.$$

Theorem 1.3 guarantees that in the definition of $\lambda_1(H)$ as well, the supremum is a maximum. We apply Theorem 1.1 to measure preserving systems.

Definition. A set $H \subset \mathbf{N}$ is a *Poincaré set* if for every measure preserving transformation T of a probability space $\langle X, \beta, \upsilon \rangle$ and for all $A \subset X$ satisfying $\upsilon(A) > 0$, there exists $h \in H$ such that $\upsilon(T^{-h}A \cap A) > 0$.

Furstenberg [7], [8] defined Poincaré sets and proved that if a nonzero polynomial P satisfies $0 \in P(\mathbf{Z}) \subset \mathbf{Z}$, the set $\mathbf{N} \cap P(\mathbf{Z})$ is a Poincaré set. This is a consequence of Example II above and the following proposition, also proved by Bertrand-Mathis [1]:

Proposition 1.4. *If H is an* FC^+*-set then H is a Poincaré set.*

Proof. A measure preserving transformation T of a probability space $\langle X, \beta, \upsilon \rangle$ induces an isometry U of $L^2(X, \beta, \upsilon)$ defined by $U(f)(x) = f(T(x))$. If $A \subset X$ has a positive measure, $\langle 1_A, 1_X \rangle = \upsilon A > 0$ and since $1_X \in \mathrm{Ker}(U - I)$, Theorem 1.1 implies that for some $h \in H$, $0 \neq \langle U^h 1_A, 1_A \rangle = \upsilon(T^{-h}A \cap A)$. $\square$

2. Summability methods and Banach limits

In this section we summarize some results from the theory of summability that will be needed in the sequel.

Definitions. Let l^∞ denote the space of bounded complex sequences $\{y_n\}_{n=1}^\infty$, with the supremum norm.

I. A *summability method* is a continuous linear functional ϕ defined on a *subspace* Dom(ϕ) of l^∞. For instance, *Cesàro summability* is the functional ϕ defined by

$$\phi(y) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} y_n,$$

and *Abel summability* is defined by

$$\psi(y) = \lim_{\substack{r\to 1\\ r<1}} (1-r) \sum_{n=1}^{\infty} y_n r^{n-1}$$

whenever the limits exist. We shall be concerned only with *regular* summability methods, i.e., methods extending ordinary convergence.

II. Let $A = (a_{mn})$ be an infinite matrix satisfying the Toeplitz conditions

$$\sup_m \sum_{n=1}^{\infty} |a_{mn}| < \infty, \quad \lim_{m\to\infty} \sum_n a_{mn} = 1, \quad \lim_{m\to\infty} a_{mn} = 0.$$

A induces a regular method ϕ_A, defined by

$$\phi_A(y) = \lim_{m\to\infty} \sum_n a_{mn} y_n = \lim(Ay)$$

whenever the limit exists.

III. The matrix A is called *positive strongly regular* if, in addition to the Toeplitz conditions, it satisfies

$$a_{mn} \geqq 0, \quad \lim_m \sum_n |a_{mn} - a_{m,n+1}| = 0,$$

IV. The sequence $\{y_n\}$ is *almost convergent* to the value ξ if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} y_{k+n} = \xi$$

holds uniformly in $k = 1, 2, \ldots$.

V. A *Banach limit* is a positive, normed, shift invariant linear functional defined on all l^∞.

LORENTZ'S THEOREM. *For $y \in l^\infty$, the following are equivalent:*

(i) *y is almost convergent to zero.*

(ii) *Every Banach limit L satisfies $L(y) = 0$.*

(iii) *Every positive strongly regular matrix A satisfies* $\lim(Ay) = 0$.

For proof, see Lorentz [14] or Petersen [17].

Lorentz's theorem is in line with our theme of studying collections of Banach limits. So is the following theorem.

THEOREM 2.1. *Let ϕ be either Abel summability or a matrix method ϕ_A corresponding to a positive, strongly regular matrix A, and let $y \in l^\infty$. If $L(y) = 0$ for all Banach limits L extending ϕ, then $\phi(y) = 0$.*

PROOF. We prove the theorem for Abel summability; the proof for matrix methods is similar. Assume $\phi(y) \neq 0$. Then for some increasing sequence $r_m \to 1$, we have

$$0 \neq \xi = \lim_{m \to \infty} (1 - r_m) \sum_{n=1}^{\infty} y_n r_m^{n-1}.$$

Define linear functionals ϕ_m on l^∞ by

$$\phi_m(x) = (1 - r_m) \sum_{n=1}^{\infty} x_n r_m^{n-1}.$$

Let L be a weak star cluster point of the sequence $\{\phi_m\}_{m=1}^{\infty}$. L is a Banach limit extending ϕ for which $L(y) = \xi \neq 0$ — a contradiction. □

We shall need a quantitative version of Lorentz's theorem. Fix $y \in l^\infty$.

$$\Lambda(N) = \sup_{k \geq 1} \left| \sum_{n=1}^{N} y_{k+n} \right|$$

is a subadditive function of N so

$$p_1(y) = \lim_{n \to \infty} \frac{\Lambda(N)}{N}$$

exists and equals $\inf_N(\Lambda(N)/N)$. Also define

$$p_2(y) = \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} y_n \right| .$$

THEOREM 2.2. *For all $y \in l^\infty$*
(a) $p_1(y) = \sup\{L(y) \mid L$ *is a Banach limit*$\}$,
(b) $p_2(y) \leq \sup\{L(y) \mid L$ *is a Banach limit extending Cesàro*$\}$.

PROOF. (a) Let U denote the left shift on l^∞. For any Banach limit L,

$$\forall N \in \mathbf{N} \quad |L(y)| = \left| L\left(\frac{1}{N}\sum_{n=1}^{N} U^n y\right)\right| = \left| L\left\{\frac{1}{N}\sum_{n=1}^{N} y_{n+k}\right\}_{k=1}^{\infty}\right| \leqq \frac{\Lambda(N)}{N},$$

Therefore $L(y) \leqq p_1(y)$.

To show the reverse, choose for each N an integer $k(N)$ such that

$$\Lambda(N) - \left|\sum_{n=1}^{N} y_{k(N)+n}\right| < 1$$

and define a linear functional α_N on l^∞,

$$\alpha_N(x) = \frac{1}{N}\sum_{n=1}^{N} x_{k(N)+n}.$$

Any weak star cluster point L of the sequence $\{\alpha_N\}_{N=1}^{\infty}$ is a Banach limit satisfying

$$L(y) = \lim_{N\to\infty} \frac{1}{N}\left|\sum_{n=1}^{N} y_{k(N)+n}\right| = p_1(y)$$

as desired. □

Part (b) is easy to show directly; we shall derive it from a stronger proposition.

THEOREM 2.3. *Let $y \in l^\infty$ be a sequence of* real *numbers. Define*

$$p_3(y) = \sup\{L(y) \mid L \text{ is a Banach limit extending Cesàro}\},$$

$$p_4(y) = \inf_{\omega} \sup_{n \geqq 1} \{y_n + \omega_n\},$$

where ω ranges over all real bounded sequences which are Cesàro summable to 0.

Finally, denote

$$p_5(y) = \sup_{\varepsilon>0} \limsup_{N\to\infty} (\varepsilon N)^{-1} \sum_{n=N}^{N(1+\varepsilon)} y_n.$$

Then

(I) $p_3 = p_4 = p_5$.

(II) *The supremum in the definition of p_3 is actually a maximum. The infimum in the definition of p_4 is a minimum, and* $\sup_{\varepsilon>0}$ *in the definition of p_5 can be replaced by* $\lim_{\varepsilon\to 0, \varepsilon>0}$.

(III) *For any complex sequence $z \in l^\infty$,*

$$\sup\{|L(z)|: L \textit{ is a Banach limit extending Cesàro}\}$$
$$= \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \limsup_{N\to\infty} (\varepsilon N)^{-1}\left|\sum_{N}^{N(1+\varepsilon)} z_n\right| .$$

Proof of I. The case $p_3 \leqq p_4$ is immediate.

For $p_4 \leqq p_5$, it suffices to show $p_5(y) < 0 \Rightarrow p_4(y) \leqq 0$. We assume $p_5(y) < 0$. Define an increasing sequence $\{N_k\}$ by

$$N_1 = 1, \qquad N_{k+1} = \min\left\{M > N_k \,\middle|\, \sum_{N_k+1}^{M} y_n < 0\right\}.$$

The assumption $p_5(y) < 0$ implies that

$$\lim_{k\to\infty} \frac{N_{k+1}}{N_k} = 1.$$

Define a real sequence $\omega \in l^\infty$ by

$$N_k \leqq n < N_{k+1} \Rightarrow \omega_n = (N_{k+1} - N_k)^{-1} \sum_{N_k+1}^{N_{k+1}} y_i - y_n.$$

It is easily seen that

$$\omega_n \xrightarrow{\text{Cesàro}} 0$$

and therefore

$$p_4(y) \leqq \sup_n \{y_n + \omega_n\} \leqq 0.$$

Consider the case $p_5 \leqq p_3$. For fixed $\varepsilon > 0$, the proof that

$$\limsup_{N\to\infty} (\varepsilon N)^{-1} \sum_{N}^{N(1+\varepsilon)} y_n \leqq p_3(y)$$

is identical to the proof of Theorem 2.2.

The proofs of (II) and (III) follow from examining the proof of (I). □

The functional p_5 is related to the notion of "Polya maximum density".

3. Equidistribution of sequences with respect to summability methods

Definitions. (i) Let G be a compact group, ϕ a summability method. A sequence $\{x_n\} \subset G$ is called *ϕ-equidistributed* if

$$(*) \qquad \forall f \in C(G), \qquad \phi\{f(x_n)\}_{n=1}^{\infty} = \int_G f d\mu$$

where μ denotes Haar measure. The Weyl criterion applies:

It is sufficient to check (*) for f a *character* of G.

(ii) The sequence $\{x_n\} \subset G$ is called *well distributed* if it is equidistributed with respect to almost convergence, and *equidistributed* if it is equidistributed with respect to Cesàro summability.

(iii) Let Φ denote the following collection of summability methods:

$$\Phi = \{\phi_A \mid A \text{ a positive strongly regular matrix}\} \cup \{\text{Abel summability}\}$$
$$\cup \{\text{almost convergence}\}.$$

Note that Φ includes Cesàro summability.

Lorentz's theorem and Theorem 2.1 imply the following

Proposition 3.1. *Let G be a compact group, and $\phi \in \Phi$. A sequence $\{x_n\} \subset G$ is ϕ-equidistributed iff it is L-equidistributed for every Banach limit L extending ϕ.*

The inner product space corresponding to a Banach limit L

Define a bilinear form $\langle \ , \ \rangle$ on l^∞ by

$$\langle w, v \rangle = L\{w_n \bar{v}_n\}_{n=1}^{\infty}.$$

We denote $J_L = \{v \in l^\infty \mid L\{|v_n|^2\}_n = 0\}$; note that $J_L \subset \operatorname{Ker} L$. The quotient space $\Omega_L = l^\infty / J_L$ is an inner product space.

Douglas [6] suggests an alternative approach to Ω_L: The Stone–Čech compactification of the positive integers is denoted $\beta\mathbf{N}$; l^∞ is isometrically isomorphic to $C(\beta\mathbf{N})$, so L can be viewed as a positive linear functional on $C(\beta\mathbf{N})$. The Riesz representation theorem attaches a measure υ_L to this functional. The Hilbert space $L^2(\upsilon_L)$ is a completion of Ω_L.

We shall not use this approach in the sequel.

Recall that we denoted the shift of l^∞ by U. J_L is a U-invariant subspace so U acts naturally on Ω_L and is an isometry there.

A Banach-limit Van der Corput Theorem

Lemma 3.2. *Let L be a Banach limit, $y \in l^\infty$ and $H \subset \mathbf{N}$ be an* FC$^+$*-set. If $\forall h \in H$, $L\{y_{n+h}\bar{y}_n\}_{n=1}^{\infty} = 0$ then also $L(y) = 0$.*

Proof. Using the space Ω_L and the isometry U induced by the shift, we can write the hypothesis

$$\forall h \in H, \qquad \langle U^h y, y \rangle = 0.$$

The characterization of FC$^+$-sets (Theorem 1.1) gives $y \perp \mathrm{Ker}(U - I)$. The constant sequence $\mathbf{1} = (1, 1, \ldots)$ is in the kernel of $U - I$, and thus

$$0 = \langle y, \mathbf{1} \rangle = L(y). \qquad \square$$

Theorem 3.3. *Let G be a compact Abelian group, $\{x_n\} \subset G$, $H \subset \mathbf{N}$ an FC$^+$-set and ϕ one of the following summability methods:*

(a) *a Banach limit,*

(b) *almost convergence,*

(c) *a positive strongly regular summability matrix,*

(d) *Abel summability.*

If for each $h \in H$ the sequence $\{x_{n+h} - x_n\}_n$ is ϕ-equidistributed, then $\{x_n\}$ is ϕ-equidistributed.

Proof. (a) First assume $\phi = L$ is a Banach limit. We must show that every nontrivial character χ of G satisfies

$$L\{\chi(x_n)\}_{n=1}^{\infty} = \int_G \chi = 0.$$

Lemma 3.2 implies the desired conclusion.

Parts (b), (c), (d) follow from (a) and Proposition 3.1. $\square$

Theorem 3.3 for Cesàro summability was proved by Kamae and Mendes France [10]. Reference to other summability methods (but not to FC$^+$-sets) can be found in Kuipers and Niederreiter [13], Cigler [4] and Kemperman [12].

Strengthening the conclusion

Let L be a Banach limit. We denote by AP(L) the subspace of Ω_L composed of L-almost periodic sequences which we define by

$$\mathrm{AP}(L) = \overline{\mathrm{span}}\{\{\zeta^n\}_{n=1}^{\infty} \mid |\zeta| = 1\}.$$

AP(L) is the closed span in Ω_L of the collection of characters of the semigroup $\mathbf{N}$.

LEMMA 3.4. *Let L be a Banach limit, H an* FC^+*-set and $y \in l^\infty$ a sequence satisfying*

$$\forall h \in H, \qquad L\{y_{n+h}\bar{y}_n\}_{n=1}^{\infty} = 0.$$

Then $y \perp \mathrm{AP}(L)$ in Ω_L.

PROOF. It suffices to check that $y \perp \{\zeta^n\}$ when $|\zeta| = 1$. This is an immediate consequence of Corollary 1.2. □

DEFINITIONS.

$$\mathrm{AP} \equiv \{\omega \in l^\infty \mid \omega + J_L \in \mathrm{AP}(L) \text{ for all Banach limits } L\},$$

$$\mathrm{AP(Cesàro)} \equiv \{\omega \in l^\infty \mid \omega + J_L \in \mathrm{AP}(L) \text{ for all Banach limits } L \text{ extending Cesàro}\},$$

EXAMPLE. Let $f\colon \mathbf{T}^d \to \mathbf{R}$ be Riemann integrable, $\alpha_1, \ldots, \alpha_d \in [0, 1)$. The sequence $z_n = f(\exp(2\pi i n \alpha_1), \ldots, \exp(2\pi i n \alpha_d))$ satisfies $z \in \mathrm{AP}$.

Preliminaries

Let G be a compact Abelian group, $\{x_n\} \subset G$, $H \subset \mathbf{N}$ an FC^+-set, and $\{m_n\}_{n=1}^{\infty} \subset \mathbf{N}$. Denote $z(k) = |\{n \mid m_n = k\}|$. We require that the indicator sequence z be bounded, but note that repetitions in $\{m_n\}$ are allowed.

Part (ii) of the following corollary appears in Daboussi and Mendes France [5].

COROLLARY 3.5. *Using the above notation*:

(i) *If* $\sup_n |m_n - m_{n-1}| < \infty$, $z \in \mathrm{AP}$, *and for each $h \in H$ the sequence $\{x_{n+h} - x_n\}_n$ is well distributed in G, then $\{x_{n_m}\}$ is also well distributed.*

(ii) *If* $\sup_n(m_n/n) < \infty$, $z \in \mathrm{AP(Cesàro)}$, *and for each $h \in H$ the sequence $\{x_{n+h} - x_n\}_n$ is equidistributed, then $\{x_{m_n}\}$ is equidistributed.*

PROOF. We prove only (ii) — the proof of (i) is similar. For each nontrivial character χ of G, we use Lemma 3.4 for the sequences $y = \chi(x_n)$ and $z \in \mathrm{AP}(L)$, where L extends Cesàro. We find that

$$\lim_{n\to\infty} \frac{1}{m_n} \sum_{k=1}^{n} \chi(x_{m_k}) = 0$$

and since $\sup(m_n/n) < \infty$, also

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi(x_{m_k}) = 0. \qquad □$$

EXAMPLE. Let $\alpha, \beta > 0$ be irrational. In Kuipers and Niederreiter [13], Chap. 5, Theorem 1.8 it is shown that the sequence $\{\lfloor n\alpha \rfloor \beta\}_{n=1}^{\infty}$ is equidistributed modulo 1 iff $1, \alpha, \alpha\beta$ are linearly independent over the rationals. (A nice proof of this fact, using ergodic theory, was given by I. Oren.) However, Corollary 3.5 implies that for all $\beta > 0$ irrational and $\alpha > 0$, the sequence $\{\lfloor n\alpha \rfloor^2 \beta\}_{n=1}^{\infty}$ is equidistributed mod 1.

Cigler considers *equidistributed sequences of measures*.

DEFINITIONS. Let G be a compact Abelian group, and let ϕ be a summability method.

(I) A sequence $\{\sigma_k\}_{k=1}^{\infty}$ of probability measures on G is ϕ-equidistributed if for every $f \in C(G)$

$$\phi \left\{ \int_G f d\sigma_k \right\}_{k=1}^{\infty} = \int_G f d\mu$$

where μ is Haar measure.

(II) Given measures υ, τ on G, define a measure $\upsilon \ominus \tau$ by:

$$\forall f \in C(G), \qquad \int_G f d(\upsilon \ominus \tau) = \int_G \int_G f(x-y) d\upsilon(x) d\tau(y).$$

PROPOSITION 3.6. *Let ϕ be one of the summability methods* (a)–(d) *appearing in the statement of Theorem* 3.3. *If $\{\sigma_k\}$ is a sequence of probability measures on the compact abelian group G such that $\{\sigma_{k+h} \ominus \sigma_k\}_{k=1}^{\infty}$ is ϕ-equidistributed for every h in an* FC$^+$*-set $H \subset \mathbf{N}$, then necessarily $\{\sigma_k\}$ is also ϕ-equidistributed.*

PROOF. Analogous to Theorem 3.3, which is the special case of Proposition 3.6 corresponding to point measures. □

4. Van der Corput sets

Theorem 3.3 motivates the following

DEFINITION. Let G be a compact Abelian group, ϕ a regular summability method. A set H of positive integers is called a *VDC set with respect to G, ϕ* if for any sequence $\{x_n\} \subset G$ such that $\{x_{n+h} - x_n\}_n$ is ϕ-equidistributed for all $h \in H$, necessarily $\{x_n\}$ is ϕ-equidistributed.

I. Z. Ruzsa [19] proved that the notions of *VDC set, with respect to T, Cesàro* and *FC⁺-set* are equivalent. In fact he proved a quantitative version of this equivalence. Our methods allow us to extend some of his results.

For $H \subset \mathbf{N}$ and $y \in l^\infty$ satisfying.

$$y_{n+h}\bar{y}_n \xrightarrow[n\to\infty]{\text{Cesàro}} 0 \qquad \text{for all } h \in H,$$

and $\| y \|_\infty \leqq 1$, Ruzsa proves

$$\limsup_{N\to\infty} \frac{1}{N}\left| \sum_{n=1}^{N} y_n \right| \leqq \sqrt{\lambda(H)}.$$

Theorem 4.1. *Let $H \subset \mathbf{N}$ and $y \in l^\infty$ with $\| y \|_\infty \leqq 1$.*

(a) *If*

$$y_{n+h}\bar{y}_n \xrightarrow[n\to\infty]{\text{Cesàro}} 0 \qquad \textit{for all } h \in H,$$

then

$$\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \limsup_{N\to\infty} (\varepsilon N)^{-1}\left| \sum_{N}^{(1+\varepsilon)N} y_n \right| \leqq \sqrt{\lambda(H)}.$$

(b) *If $\{y_{n+h}\bar{y}_n\}$ is almost convergent to 0 for all $h \in H$, then*

$$\limsup_{N} \sup_{k} \frac{1}{N}\left| \sum_{n=1}^{N} y_{n+k} \right| \leqq \sqrt{\lambda(H)}.$$

Proof. (a) By Theorem 2.3(III) it suffices to show that for any Banach limit L extending Cesàro, $|L(y)| \leqq \sqrt{\lambda(H)}$.

Let U be the shift on Ω_L and $\mathbf{1} = (1, 1, 1, \ldots)$. The hypotheses $U^h y \perp y$ $\forall h \in H$, $\| y \|_{\Omega_L} \leqq \| y \|_\infty \leqq 1$ imply, using Theorem 1.3, that

$$|L(y)|^2 = |\langle y, \mathbf{1} \rangle|^2 \leqq \lambda_1(H) = \lambda(H).$$

(b) Follows similarly from Theorem 2.2. □

Remark. In Theorem 4.1, the boundedness assumption on y can be weakened (as done by Ruzsa in [19]).

Van der Corput sets with respect to other groups

Question. For which compact Abelian groups G are all VDC set with respect to G, Cesàro necessarily FC⁺-sets? (The reverse implication always holds, by Theorem 3.3.)

We cannot presently answer this question, but an extension of Ruzsa's result in this direction is given.

THEOREM 4.2. *Let G_0 be a compact Abelian metrizable group. If G is a direct sum $G_0 \oplus \mathbf{T}$ then any VDC set with respect to G, Cesàro is an* FC^+*-set.*

We shall need a lemma.

LEMMA 4.3. *Let G_0 be an Abelian metrizable compact group, and let $G = G_0 \oplus \mathbf{T}$.*

(a) *Let $\{t_n\}_{n=1}^{\infty} \subset \mathbf{T}$, $\{y_n\}_{n=1}^{\infty} \subset G_0$. If $\{y_n + t_n\}$ is equidistributed in G, then $\{t_n\}$ is equidistributed in $\mathbf{T}$.*

(b) *Let $\{\theta_n\}$ be an equidistributed sequence modulo 1. For almost all sequences $\{x_n\} \subset G_0$ with respect to Haar measure on $G_0^{\mathbf{N}}$, $\{x_n + \theta_n\}$ is equidistributed in G. Moreover, for almost all $\{x_n\} \in G_0^{\mathbf{N}}$, $\{x_{n+h} - x_n + \theta_n\}_n$ is equidistributed in G for all $h \in \mathbf{N}$.*

PROOF. (a) is immediate.

(b) The dual group of G is countable, so it suffices to show that for each nontrivial character χ of G,

$$\lim_N \frac{1}{N} \sum_{n=1}^{N} \chi(x_n + \theta_n) = 0 \qquad \text{for almost all } \{x_n\} \in G_0^{\mathbf{N}}.$$

If $\chi(G_0 \oplus \{0\}) = \{1\}$ this follows from the hypothesis on $\{\theta_n\}$; otherwise, it follows from the strong law of large numbers. The sequence $x_{n+h} - x_n + \theta_n$ is dealt with similarly. □

PROOF OF THEOREM 4.2. Let H be a VDC set with respect to G, Cesàro. We show that H is a VDC set with respect to $\mathbf{T}$, Cesàro. Let $\{t_n\} \subset \mathbf{T}$ be such that $\{t_{n+h} - t_n\}_n$ is equidistributed for all $h \in H$.

By the lemma, for almost all $\{x_n\} \in G_0^{\mathbf{N}}$ and all $h \in H$, $\{x_{n+h} - x_n + t_{n+h} - t_n\}_n$ is equidistributed in G, so $\{x_n + t_n\}$ is equidistributed by hypothesis.

By the lemma $\{t_n\}$ itself is equidistributed. □

For further related results, see Bourgain [2] and Peres [16].

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